VALSCALC

Chapter 3 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 3: Applications of Differentiation.

3.1: Minimum and Maximum Values
3.2: Mean Value Theorem and Rolle's Theorem
3.3: Using the First and Second Derivatives
3.4: Particle Motion
3.5: Indeterminate Forms and L'Hopital's Rule
3.6: Optimization
3.7: Applications of Differentiation in Economics
3.8: Newton's Method

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Applications of Differentiation

Number of Questions—14

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points	Points Available
Short Questions		55
Question 12		15
Question 13		15
Question 14		15
TOTAL		**100**

Short Questions

1. Evaluate $\lim_{x\to 0} \dfrac{x^2 - \sin 3x}{\tan x}$. *(5 pts.)*

2. Find the absolute minimum and absolute maximum values of $f(x) = x^3 - 3x^2 + 3x + 10$ on $[0, 2]$. *(5 pts.)*

3. Use Newton's Method with two iterations, starting with the initial approximation $x_1 = 0$, to estimate the negative zero of $f(x) = 2x^2 - 2x - 4$. *(5 pts.)*

4. Find all the values of c that satisfy the Mean Value Theorem (if it is applicable) for $f(x) = x^3 + 4x - 6$ on $-1 \leqslant x \leqslant 1$. *(5 pts.)*

5. Determine the x-coordinates of all the inflection points on the graph of *(5 pts.)*

$$y = x^4 - 4x^3 - 18x^2 + 16x - 2\,.$$

6. Calculate the area of the largest rectangle that can be inscribed within the graph of $y = 2 - x^2$, $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$. *(5 pts.)*

7. Find all the critical numbers of $g(x) = \frac{2}{3}x^3 + \frac{32}{x}$. Use the Second-Derivative Test to determine whether g has a relative maximum, relative minimum, or neither at each critical number. *(5 pts.)*

8. On what intervals is the graph of $y = e^{-2x} - e^{4x}$ concave down? *(5 pts.)*

9. What point on the line $y = 5 - x$ is closest to the point $(4,2)$? *(5 pts.)*

10. Calculate $\lim_{x\to\infty} (\ln x)^{1/x^2}$. *(5 pts.)*

11. As $x \to \infty$, which function grows faster: $f(x) = x^3 + 4x^2 + 8$ or $g(x) = \dfrac{x^6 - 4}{x}$? Justify your answer using limits. *(5 pts.)*

Long Questions

12. A particle travels along the x-axis with time $t \geqslant 0$ according to the position function $x(t) = 5 - \cos^2 t$.

(a) On $(0, \pi)$, find all the particle's turning points. *(4 pts.)*

(b) What is the total distance the particle travels on $[0, \pi]$? *(3 pts.)*

(c) On $(0, \pi)$, determine when the particle is speeding up and when it is slowing down. *(6 pts.)*

(d) Does the particle ever reach a final destination? Justify your answer. *(2 pts.)*

13. A website sells hard drives for $60 each in a high-demand market. The business's cost in manufacturing x hard drives follows the function $C(x) = \frac{1}{4}x^2 + 20x + 500$.

(a) Determine the website's marginal cost function and average cost function. (2 pts.)

(b) Show that the website's profit function is $P(x) = -\frac{1}{4}x^2 + 40x - 500$. (3 pts.)

(c) What is the maximum profit the website can generate? (3 pts.)

(d) The breakeven point is the quantity produced such that the company earns 0 profit—that is, when $P(x) = 0$. Identify an initial starting approximation x_1 such that Newton's Method fails to approximate the breakeven point. *(2 pts.)*

(e) A competing store sells solid-state drives, selling 40 per month when they are priced at \$50 each. The manager predicts that the store will sell 2 fewer solid-state drives per month for every \$5 the price increases. To maximize monthly revenue, how much should the store charge per solid-state drive? *(5 pts.)*

14. Let $f(x) = 3x^{5/3} - 15x^{2/3}$. It can be shown that $f'(x) = 5x^{2/3} - 10x^{-1/3}$.

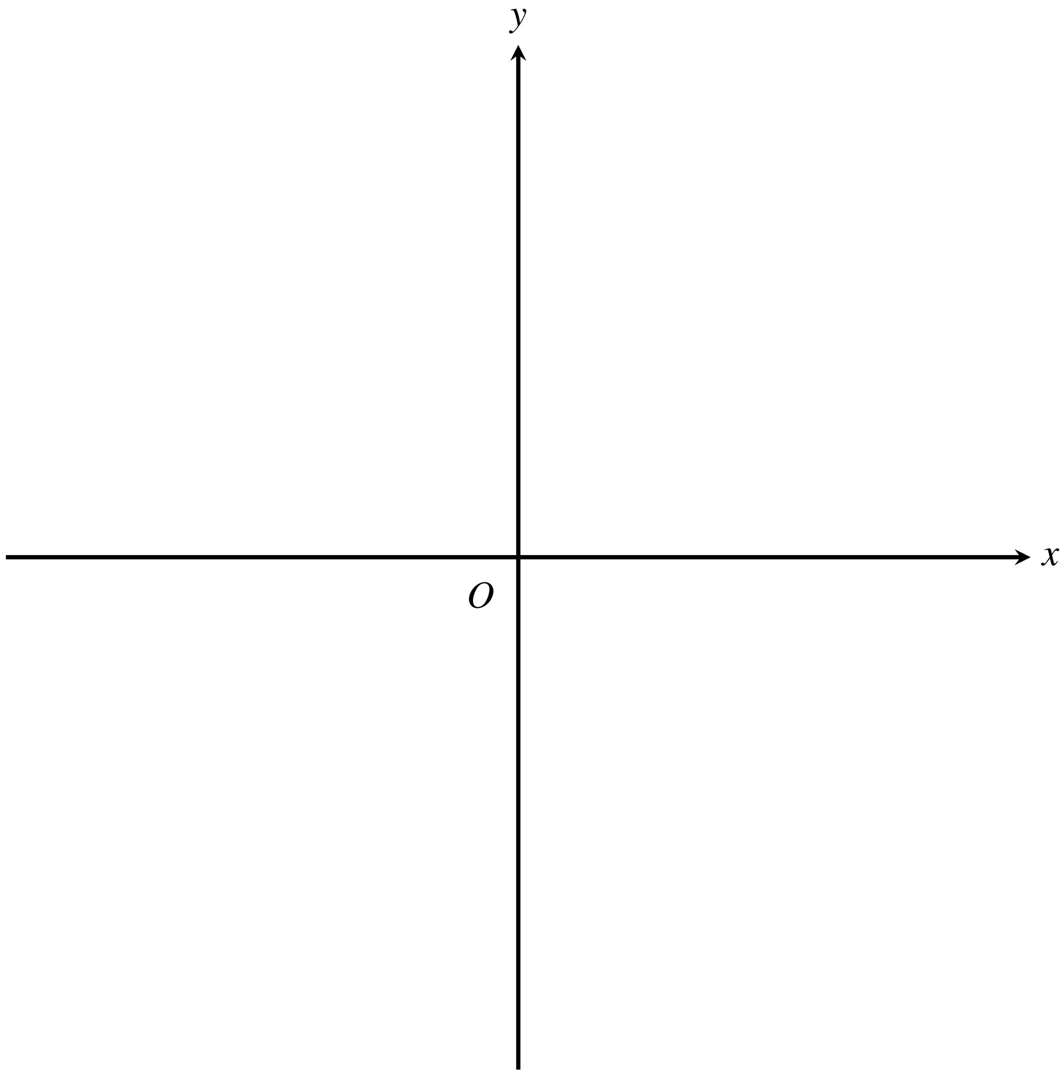
(a) Find all the critical numbers of f. On what open intervals is f increasing, and on what open intervals is f decreasing? *(3 pts.)*

(b) Using the First-Derivative Test, determine whether each critical number is the location of a relative maximum, relative minimum, or neither of f. *(2 pts.)*

(c) On what intervals is the graph of $y = f(x)$ concave up, and on what intervals is it concave down? *(4 pts.)*

(d) Determine the intercepts of the graph of $y = f(x)$. *(1 pt.)*

(e) Sketch the graph of $y = f(x)$ in the following figure. *(5 pts.)*



This marks the end of the test. The solutions and scoring rubric begin on the next page.

Short Questions (5 points each)

1. The numerator and denominator both approach 0 as $x \to 0$, so the limit is in the indeterminate form $\frac{0}{0}$. Therefore, by L'Hôpital's Rule,

$$\lim_{x\to 0} \frac{x^2 - \sin 3x}{\tan x} = \lim_{x\to 0} \frac{\frac{d}{dx}(x^2 - \sin 3x)}{\frac{d}{dx}(\tan x)} \quad *$$

$$= \lim_{x\to 0} \frac{2x - 3\cos 3x}{\sec^2 x} \quad *$$

$$= \frac{2(0) - 3\cos 0}{\sec^2 0} \quad *$$

$$= \boxed{-3} \quad **$$

2. Differentiating gives

$$f'(x) = 3x^2 - 6x + 3. \quad *$$

The critical numbers are found by solving $f'(x) = 0$:

$$3x^2 - 6x + 3 = 0 \quad *$$

$$3(x-1)^2 = 0$$

$$\implies x = 1. \quad *$$

Using the Closed Interval Method, we evaluate f at the critical number 1 and at the endpoints, 0 and 2. Doing so shows

$$f(0) = 10 \qquad f(1) = 11 \qquad f(2) = 12.$$

Hence,

$$\text{absolute minimum: } \boxed{10} \quad *$$

$$\text{absolute maximum: } \boxed{12} \quad *$$

3. The derivative function is

$$f'(x) = 4x - 2. \quad *$$

The first iteration of Newton's Method with $x_1 = 0$ gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0 - \frac{-4}{-2} \quad *$$

$$= -2\,. \quad *$$

The second iteration, with $x_2 = -2$, is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -2 - \frac{8}{-10} \quad *$$

$$= \boxed{-1.2} \quad *$$

4. The Mean Value Theorem is applicable because $f(x) = x^3 + 4x - 6$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$. On $[-1,1]$, the average rate of change of f is

$$\frac{f(1) - f(-1)}{1-(-1)} = \frac{-1-(-11)}{2} = 5\,. \quad *$$

Differentiating f produces

$$f'(x) = 3x^2 + 4\,. \quad *$$

By the Mean Value Theorem, a value c exists in $(-1,1)$ such that $f'(c) = 5$:

$$3c^2 + 4 = 5 \quad *$$

$$c^2 = \frac{1}{3}$$

$$\implies c = \boxed{\frac{1}{\sqrt{3}}}, \boxed{-\frac{1}{\sqrt{3}}} \quad **$$

5. Differentiating gives

$$y' = 4x^3 - 12x^2 - 36x + 16\,,$$

$$y'' = 12x^2 - 24x - 36\,. \quad *$$

We solve $y'' = 0$ as follows:

$$12x^2 - 24x - 36 = 0 \quad *$$

$$12(x-3)(x+1) = 0$$

$$\implies x = -1\,, 3\,. \quad *$$

At $x = -1$, y'' changes sign from positive to negative; at $x = 3$, y'' changes sign from negative to positive. Hence, the x-coordinates of the inflection points are

$$x = \boxed{-1} \qquad \text{and} \qquad x = \boxed{3} \quad **$$

6. Consider a rectangle inscribed under the curve. By modeling with $x > 0$, the rectangle's width is $2x$ and its height is $y = 2 - x^2$. Thus, the rectangle's area is

$$A(x) = 2x(2 - x^2) = 4x - 2x^3\,. \quad *$$

Differentiating gives

$$A'(x) = 4 - 6x^2\,.$$

The critical numbers of A are attained by solving $A'(x) = 0$, as follows:

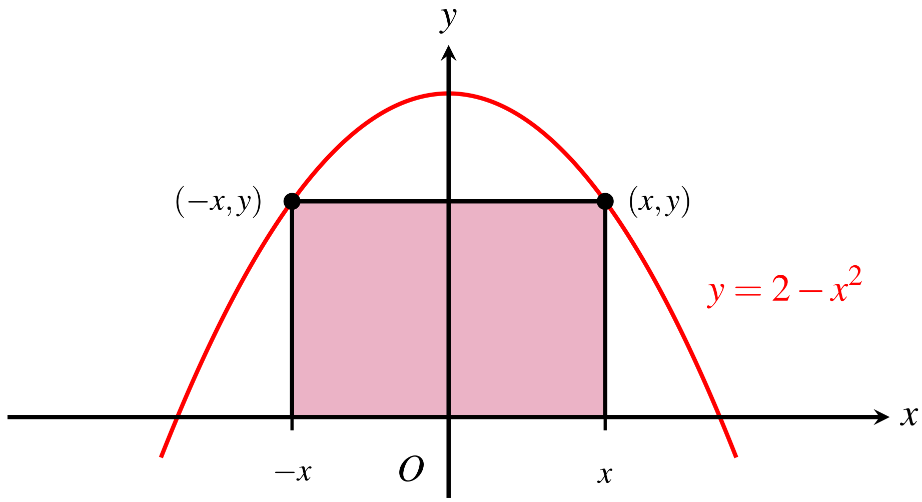
$$4 - 6x^2 = 0 \quad *$$

$$\implies x = \frac{2}{\sqrt{6}}\,. \quad *$$

This critical number is the location of the absolute maximum of $A(x)$. (Because we seek $x > 0$, we ignore the negative solution.) At this value, the area is

$$A\left(\frac{2}{\sqrt{6}}\right) = 4\left(\frac{2}{\sqrt{6}}\right) - 2\left(\frac{2}{\sqrt{6}}\right)^3 \quad *$$

$$= \boxed{\frac{16}{3\sqrt{6}}} \quad *$$



7. Differentiating yields

$$g'(x) = 2x^2 - \frac{32}{x^2}.$$

We find the critical numbers by solving $g'(x) = 0$:

$$2x^2 - \frac{32}{x^2} = 0 \quad *$$

$$x^4 - 16 = 0$$

$$\implies x = \pm 2. \quad *$$

The second derivative is

$$g''(x) = 4x + \frac{64}{x^3}. \quad *$$

Evaluating g'' at each critical number, we see

$$g''(-2) = -16 < 0\,,$$

$$g''(2) = 16 > 0\,.$$

Thus,

$$x = -2: \boxed{\text{relative maximum}}$$ *

$$x = 2: \boxed{\text{relative minimum}}$$ *

8. We take derivatives to find

$$y' = -2e^{-2x} - 4e^{4x}\,,$$ *

$$y'' = 4e^{-2x} - 16e^{4x}\,.$$ *

The graph is concave down when $y'' < 0$, so we see

$$4e^{-2x} - 16e^{4x} < 0$$ *

$$4(e^{-2x} - 4e^{4x}) < 0$$

$$4e^{4x} > e^{-2x}$$

$$e^{6x} > \tfrac{1}{4}$$

$$\boxed{x > \tfrac{1}{6}\ln\tfrac{1}{4}}$$ **

9. The distance between $(x,y) = (x, 5-x)$ and the point $(4,2)$ is, by the Distance Formula,

$$d = \sqrt{(x-4)^2 + (y-2)^2}$$ *

$$= \sqrt{(x-4)^2 + (3-x)^2}\,.$$ *

The critical numbers of d^2 are the same as the critical numbers of d, so it is easier to consider d^2. Letting

$q(x) = d^2$, we have

$$q(x) = (x-4)^2 + (3-x)^2,$$

$$q'(x) = 2(x-4) - 2(3-x) = 4x - 14.$$ *

Solving $q'(x) = 0$ shows

$$4x - 14 = 0 \implies x = 3.5.$$ *

Then

$$y = 5 - 3.5 = 1.5.$$

Hence, the distance is minimized at the point

$$\boxed{(3.5, 1.5)}$$ *

10. As $x \to \infty$, $\ln x \to \infty$ and $\frac{1}{x^2} \to 0$. Thus, this limit is in the indeterminate form ∞^0. Letting $y = (\ln x)^{1/x^2}$, we attain

$$\ln y = \frac{1}{x^2}\ln(\ln x) = \frac{\ln(\ln x)}{x^2}.$$ *

Now, using L'Hôpital's Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(\ln x)}{x^2}$$ *

$$= \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{2x}$$ *

$$= \lim_{x\to\infty} \frac{1}{2x^2 \ln x}$$

$$= 0.$$ *

It then follows that

$$\lim_{x\to\infty} (\ln x)^{1/x^2} = e^{\lim_{x\to\infty} \ln y} = e^0 = \boxed{1}$$ *

11. We consider a ratio of f and g—say,

$$\frac{f(x)}{g(x)} = \frac{x^3+4x^2+8}{\frac{x^6-4}{x}} = \frac{x^4+4x^3+8x}{x^6-4}\,. \quad *$$

Then

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{f(x)}{g(x)} \quad * \\ &= \lim_{x\to\infty} \frac{x^4+4x^3+8x}{x^6-4} \\ &= 0\,. \quad * \end{aligned}$$

Because $L = 0$,

$$\boxed{g(x)} \text{ grows faster}\,. \quad **$$

Long Questions (15 points each)

12. (a) The particle has turning points when the velocity $v(t)$ changes sign. Velocity is the derivative of position, so

$$\begin{aligned} v(t) &= x'(t) & * \\ &= 2\cos t \sin t & * \\ &= \sin 2t\,. \end{aligned}$$

In $(0,\pi)$, the solution to $v(t)=0$ is $t=\dfrac{\pi}{2}$. At this point, $v(t)$ changes sign from positive to negative. Hence, the turning point is

$$t = \boxed{\frac{\pi}{2}}$$ **

(b) The distance traveled is the sum of magnitudes of the displacements. From $t=0$ to $t=\dfrac{\pi}{2}$, the distance traveled is

$$\left|x\left(\frac{\pi}{2}\right)-x(0)\right| = |5-4| = 1\,.$$ *

From $t=\dfrac{\pi}{2}$ to $t=\pi$, the distance traveled is

$$\left|x(\pi)-x\left(\frac{\pi}{2}\right)\right| = |4-5| = 1\,.$$ *

Accordingly, the total distance traveled on $[0,\pi]$ is

$$\left|x\left(\frac{\pi}{2}\right)-x(0)\right| + \left|x(\pi)-x\left(\frac{\pi}{2}\right)\right| = 1+1 = \boxed{2}$$ *

(c) Let $a(t)$ be the acceleration function. The particle is speeding up when $v(t)$ and $a(t)$ have the same sign, and it is slowing down when they have opposite signs. We have

$$a(t) = v'(t) = 2\cos 2t\,.$$ *

The solutions to $a(t)=0$ in $(0,\pi)$ are

$$t=\frac{\pi}{4} \qquad \text{and} \qquad t=\frac{3\pi}{4}\,.$$ *

We consider subintervals in $(0,\pi)$ whose endpoints contain the zeros of $v(t)$ and $a(t)$—namely,

$t = \frac{\pi}{4}$, $t = \frac{\pi}{2}$, and $t = \frac{3\pi}{4}$. The following table shows the signs of $v(t)$ and $a(t)$ across these subintervals.

Interval of t	$v(t) = \sin 2t$	$a(t) = 2\cos 2t$
$\left(0, \frac{\pi}{4}\right)$	$+$	$+$
$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$	$+$	$-$
$\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$	$-$	$-$
$\left(\frac{3\pi}{4}, \pi\right)$	$-$	$+$

Thus,

$$\text{speeding up: } \boxed{\left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)}$$ **

$$\text{slowing down: } \boxed{\left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right)}$$ **

(d) The limit

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} (5 - \cos^2 t)$$

does not exist. $\boxed{\text{No}}$, the particle does not reach a final destination. **

13. (a) The marginal cost function is

$$C'(x) = \boxed{\frac{1}{2}x + 20}$$ *

The average cost function is

$$\overline{C}(x) = \frac{C(x)}{x} = \boxed{\frac{1}{4}x + 20 + \frac{500}{x}}$$ *

(b) The revenue function is price $\times$ quantity, or

$$R(x) = 60x.$$ *

Profit is the difference between revenue and cost:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 60x - \left(\frac{1}{4}x^2 + 20x + 500\right) \\ &= \boxed{-\frac{1}{4}x^2 + 40x - 500} \end{aligned}$$ * *

(c) First we differentiate $P(x)$:

$$P'(x) = -\frac{1}{2}x + 40\,.$$ *

Then the critical numbers of P are found by solving $P'(x) = 0$:

$$-\frac{1}{2}x + 40 = 0 \implies x = 80\,.$$ *

This value corresponds to the absolute maximum of $P(x)$. Accordingly, the maximum profit is

$$P(80) = -\frac{1}{4}(80)^2 + 40(80) - 500 = \boxed{\$1100}$$ *

(d) Newton's Method fails when the tangent line is horizontal. Because $P'(80) = 0$, the graph of P has a horizontal tangent at $x = 80$. Consequently, Newton's Method fails for

$$x_1 = \boxed{80}$$ **

(e) Let s be the number of \$5 price increases. The revenue is price × quantity, or

$$R(s) = (50 + 5s)(40 - 2s) = 2000 + 100s - 10s^2\,.$$ **

To maximize R, we first find its critical numbers. Differentiating gives

$$R'(s) = 100 - 20s\,.$$ *

Solving $R'(s) = 0$ shows

$$100 - 20s = 0 \implies s = 5\,.$$ *

Hence, there should be 5 price increases of \$5, or $5 \times \$5 = \$25$. The store should therefore price each solid-state drive at

$$\$50 + \$25 = \boxed{\$75}$$ *

14. (a) We solve $f'(x) = 0$ as follows:

$$5x^{2/3} - 10x^{-1/3} = 0$$ *

$$5x^{-1/3}(x-2) = 0$$

$$\implies x = 2\,.$$

In addition, $f'(0)$ is undefined while $f(0) = 0$. Therefore, the critical numbers of f are

$$x = \boxed{0} \qquad \text{and} \qquad x = \boxed{2}$$ *

We see $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $0 < x < 2$, and $f'(x) > 0$ for $x > 2$. Therefore,

$$\text{increasing: } \boxed{(-\infty, 0) \cup (2, \infty)}$$

$$\text{decreasing: } \boxed{(0, 2)}$$ *

(b) The sign of f' changes from positive to negative at $x = 0$, and from negative to positive at $x = 2$. As a result,

$$x = 0\text{: } \boxed{\text{relative maximum}}$$ *

$$x = 2\text{: } \boxed{\text{relative minimum}}$$ *

(c) The second derivative is

$$f''(x) = \frac{10}{3}x^{-1/3} + \frac{10}{3}x^{-4/3}\,.$$ *

Solving $f''(x) = 0$ gives

$$\frac{10}{3}x^{-1/3} + \frac{10}{3}x^{-4/3} = 0$$
$$\frac{10}{3}x^{-4/3}(x+1) = 0$$
$$\implies x = -1\,. \quad *$$

For $x < -1$, $f''(x) < 0$; for $x > -1$, $f''(x) > 0$. Note that $x = 0$ is not an inflection point because no sign change occurs at that value. The intervals are therefore as follows:

concave up: $\boxed{(-1,0)\cup(0,\infty)}$ *

concave down: $\boxed{(-\infty,-1)}$ *

(d) To find the x-intercepts, we set $f(x) = 0$:

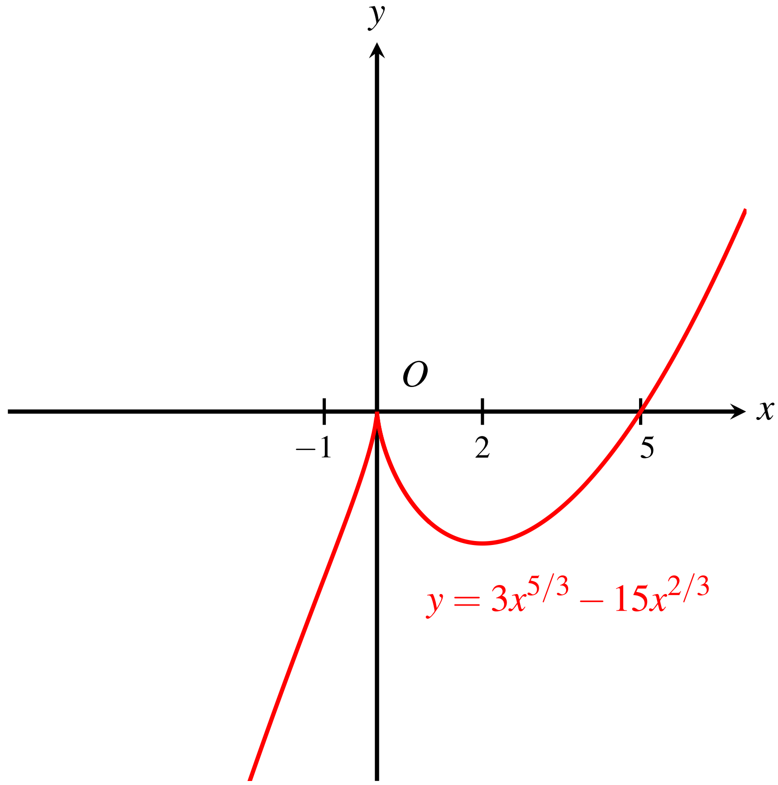
$$3x^{5/3} - 15x^{2/3} = 0$$
$$3x^{2/3}(x-5) = 0$$
$$\implies x = 0, 5\,.$$

The y-intercept is $(0,0)$. Hence, the intercepts are

$\boxed{(0,0)}$ $\boxed{(5,0)}$ *

(e) The graph of $y = f(x)$ has the following features:

- Intercepts at $(0,0)$ and $(5,0)$ *
- Increasing on $(-\infty,0)\cup(2,\infty)$ and decreasing on $(0,2)$ *
- Horizontal tangent at $x = 2$ *
- Sharp turn at $x = 0$ *
- Concave up on $(-1,0)\cup(0,\infty)$ *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O
x
−1
2
5
y = 3x^{5/3} − 15x^{2/3}